

# ISOPERIMETRIC INEQUALITY FOR THE SECOND NON-ZERO EIGENVALUE OF THE LAPLACE-BELTRAMI OPERATOR ON THE PROJECTIVE PLANE

NIKOLAI S. NADIRASHVILI AND ALEXEI V. PENSKOI

**ABSTRACT.** An isoperimetric inequality for the second non-zero eigenvalue of the Laplace-Beltrami operator on the real projective plane is proven. For a metric of area 1 this eigenvalue is not greater than  $20\pi$ . This value could be attained as a limit on a sequence of metrics of area 1 on the projective plane converging to a singular metric on the projective plane and the sphere with standard metrics touching in a point such that the ratio of the areas of the projective plane and the sphere is  $3 : 2$ . It is also proven that the multiplicity of the second non-zero eigenvalue on the projective plane is at most 6.

## 1. INTRODUCTION

Let  $M$  be a closed surface and  $g$  be a Riemannian metric on  $M$ . Let us consider the Laplace-Beltrami operator  $\Delta : C^\infty(M) \rightarrow C^\infty(M)$  associated with the metric  $g$ ,

$$\Delta f = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right),$$

and its eigenvalues

$$(1) \quad 0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \lambda_3(M, g) \leq \dots$$

Let us denote by  $m(M, g, \lambda_i)$  the multiplicity of the eigenvalue  $\lambda_i(M, g)$ , i.e. how many times the value of  $\lambda_i(M, g)$  appears in the sequence (1).

Let us consider a functional

$$\bar{\lambda}_i(M, g) = \lambda_i(M, g) \text{Area}(M, g),$$

where  $\text{Area}(M, g)$  is the area of  $M$  with respect to the Riemannian metric  $g$ . This functional is sometimes called an eigenvalue normalized by the area or simply a normalized eigenvalue.

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Yang and Yau proved in the paper [45] that if  $M$  is an orientable surface of genus  $\gamma$  then

$$\bar{\lambda}_1(M, g) \leq 8\pi(\gamma + 1).$$

Actually, the arguments of Yang and Yau imply a stronger estimate,

$$\bar{\lambda}_1(M, g) \leq 8\pi \left\lceil \frac{\gamma + 3}{2} \right\rceil,$$

see the paper [32]. Here  $\lceil \cdot \rceil$  denotes the integer part of a number.

Later Korevaar proved in the paper [27] that there exists a constant  $C$  such that for any  $i > 0$  and any compact surface  $M$  of genus  $\gamma$  the following upper bound holds,

$$\bar{\lambda}_i(M, g) \leq C(\gamma + 1)i.$$

It follows that the functionals  $\bar{\lambda}_i(M, g)$  are bounded from above and it is a natural question to find for a given compact surface  $M$  and number  $i \in \mathbb{N}$  the quantity

$$\Lambda_i(M) = \sup_g \bar{\lambda}_i(M, g),$$

where the supremum is taken over the space of all Riemannian metrics  $g$  on  $M$ .

Sometimes this problem is called the *problem of geometric optimization of eigenvalues of the Laplace-Beltrami operator*.

Let us remark that the functional  $\bar{\lambda}_i(M, g)$  is invariant under rescaling of the metric  $g \mapsto tg$ , where  $t \in \mathbb{R}_+$ . It follows that it is equivalent to the problem of finding  $\sup \lambda_i(M, g)$ , where the supremum is taken over the space of all Riemannian metrics  $g$  of area 1 on  $M$ . That's why this problem is sometimes called the *isoperimetric problem for eigenvalues of the Laplace-Beltrami operator*.

**Definition 1.1.** *Let  $M$  be a closed surface. A metric  $g_0$  on  $M$  is called maximal for the functional  $\bar{\lambda}_i(M, g)$  if*

$$\Lambda_i(M) = \bar{\lambda}_i(M, g_0)$$

If a maximal metric exists, it is defined up to multiplication by a positive constant due to the rescaling invariance of the functional.

Surprisingly, the list of known results is quite short.

Hersch proved in 1970 in the paper [19] that the standard metric on the sphere is the unique maximal metric for  $\bar{\lambda}_1(\mathbb{S}^2, g)$  and

$$\Lambda_1(\mathbb{S}^2) = 8\pi.$$

Li and Yau proved in 1982 in the paper [29] that the standard metric on the projective plane is the unique maximal metric for  $\bar{\lambda}_1(\mathbb{RP}^2, g)$  and

$$\Lambda_1(\mathbb{RP}^2) = 12\pi.$$

The first author proved in 1996 in the paper [32] that the standard metric on the equilateral torus is the unique maximal metric for  $\bar{\lambda}_1(\mathbb{T}^2, g)$  and

$$\Lambda_1(\mathbb{T}^2) = \frac{8\pi^2}{\sqrt{3}}.$$

It is not always that a maximal metric exists. As it was proved by the first author in 2002 in the paper [33],

$$\Lambda_2(\mathbb{S}^2) = 16\pi.$$

However, there is no maximal metric. The supremum is attained as a limit on a sequence of smooth metrics on the sphere converging to a singular metric on two spheres of the same radius touching in a point.

The functional  $\bar{\lambda}_i(M, g)$  depends continuously on the metric  $g$ . However, when  $\bar{\lambda}_i(M, g)$  is a multiple eigenvalue this functional is not in general differentiable. If we consider an analytic variation  $g_t$  of the metric  $g = g_0$ , then it was proved by Berger [4], Bando and Urakawa [1], El Soufi and Ilias [16] that the left and right derivatives of the functional  $\bar{\lambda}_i(M, g_t)$  with respect to  $t$  exist. This leads us to the following definition given by the first author in the paper [32] and by El Soufi and Ilias in the papers [15, 16].

**Definition 1.2.** *A Riemannian metric  $g$  on a closed surface  $M$  is called extremal metric for the functional  $\bar{\lambda}_i(M, g)$  if for any analytic deformation  $g_t$  such that  $g_0 = g$  one has*

$$\left. \frac{d}{dt} \bar{\lambda}_i(M, g_t) \right|_{t=0+} \leq 0 \leq \left. \frac{d}{dt} \bar{\lambda}_i(M, g_t) \right|_{t=0-}.$$

Jakobson, the first author and Polterovich proved in 2006 in the paper [21] that the metric on the Klein bottle realized as so called bipolar Lawson surface  $\tilde{\tau}_{3,1}$ , is extremal for  $\bar{\lambda}_1(\mathbb{KL}, g)$ . It was conjectured in this paper that this metric is the maximal one. El Soufi, Giacomini and Jazar proved in the same year 2006 in the paper [17] that this metric on  $\tilde{\tau}_{3,1}$  is the unique extremal metric for the  $\bar{\lambda}_1(\mathbb{KL}, g)$ .

In fact, it follows from the papers [35, 41] that there is a smooth with at most finite numbers of conical points metric  $g_K$  on the Klein bottle such that  $\sup \bar{\lambda}_1(\mathbb{KL}, g)$  is attained on  $g_K$ . Therefore, there is a common belief that the metric on  $\tilde{\tau}_{3,1}$  is the maximal one and, hence,

$$\Lambda_1(\mathbb{KL}) = \bar{\lambda}_1(\mathbb{KL}, g_{\tilde{\tau}_{3,1}}) = 12\pi E \left( \frac{2\sqrt{2}}{3} \right),$$

where  $E$  is the complete elliptic integral of the second kind.

More results on extremal metrics on tori and Klein bottles could be found in the papers [15, 22, 23, 24, 28, 37, 38, 40]. A review of these results is given by the second author in the paper [39].

The last result concerning maximization of eigenvalues was obtained by the first author and Sire in 2015 in the paper [36], where the equality

$$\Lambda_3(\mathbb{S}^2) = 24\pi$$

was proven. It turns out that there is no maximal metric but the supremum could be obtained as a limit on a sequence of metrics on the sphere converging to a singular metric on three touching spheres of the same radius.

It was conjectured in the paper [36] that

$$\Lambda_k(\mathbb{S}^2) = 8\pi k.$$

The main goal of the present paper is to prove the following result.

**Theorem 1.3.** *For the second non-zero eigenvalue on the projective plane one has the equality*

$$\Lambda_2(\mathbb{RP}^2) = 20\pi,$$

*where the supremum is taken over the space of all Riemannian metrics  $g$  on  $\mathbb{RP}^2$  including metrics with conical singularities. There is no maximal metric. The supremum is attained as a limit on a sequence of metrics on the projective plane converging to a singular metric on the projective plane and the sphere with standard metrics touching in a point such that the ratio of the areas of the projective plane and the sphere is 3 : 2.*

We postpone the definition of metrics with conical singularities till Section 5.

This Theorem could be stated as an isoperimetric inequality

$$\lambda_2(\mathbb{RP}^2, g) \leq 20\pi$$

for any metric  $g$  of area 1.

The paper is organized in the following way. In Section 2 we recall the relation between extremal metrics and minimal immersions into spheres and explain the importance of upper bounds on the multiplicity of eigenvalues.

In Section 3 we recall basics of the theory of nodal graphs and the Courant Nodal Domain Theorem that we need in order to obtain in Section 4 an upper bound for  $m(\mathbb{RP}^2, g, \lambda_2)$ . Let us remark that bounds on multiplicity of eigenvalues of the Laplace-Beltrami operator on surfaces were subject of numerous papers, see e.g. [6, 11, 20, 25, 31].

In Section 5 we pass from minimal immersions to harmonic immersions, extend our considerations to harmonic immersions with branch points and metrics with conical singularities and explain why the results from the previous sections also hold in this case.

In Section 6 we recall Calabi-Barbosa Theorem about harmonic immersions with branch points  $\mathbb{S}^2 \rightarrow \mathbb{S}^n$  and apply it to our situation.

Section 7 contains Bryant description of the space of harmonic immersions with branch points  $\mathbb{S}^2 \rightarrow \mathbb{S}^4$  and results about singularities of these maps.

Section 8 contains the theory of maximization of higher order eigenvalues in a conformal class. It explains so-called “bubbling phenomenon”.

Finally, in Section 9 we prove Theorem 1.3.

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## 2. EXTREMAL METRICS AND MINIMAL IMMERSIONS TO SPHERES

In this Section we recall the relation between extremal metrics and minimal immersions into spheres and explain the importance of upper bounds on the multiplicity of eigenvalues.

Let us recall the definition of a minimal map, see e.g. [13, 14].

Let  $(M, g)$  be a Riemannian manifold of dimension  $m$ . Let  $\alpha$  be a symmetric bilinear 2-form on  $TM$ . Let  $\sigma_k$  be the  $k$ -th elementary symmetric function. Let  $\sigma_k(\alpha) = \sigma_k(\lambda_1, \dots, \lambda_m)$ , where  $\lambda_i$  are eigenvalues of  $\alpha$  related to  $g$ , i.e. roots of the polynomial  $\det(\alpha_{ij} - \lambda g_{ij}) = 0$ .

**Definition 2.1.** *Let  $(M, g)$  and  $(N, h)$  be Riemannian manifolds. A smooth map  $f : M \rightarrow N$  is called minimal if  $f$  is an extremal for the volume functional*

$$V[f] = \int_M \sqrt{|\sigma_m(f^*h)|} dVol_g,$$

where  $m = \dim M$ .

It is well-known that a surface  $M \looparrowright \mathbb{R}^3$  is minimal if and only if the coordinate functions  $x^i$  are harmonic with respect to the Laplace-Beltrami operator on  $M$ . A similar result holds for a minimal submanifold in  $\mathbb{R}^n$ . Since harmonic functions are eigenfunctions with eigenvalue 0, it is natural to ask what is an analogue of this statement for a non-zero eigenvalue. The answer was given by Takahashi in 1966.

**Theorem 2.2** (Takahashi [44]). *If an isometric immersion*

$$f : M \looparrowright \mathbb{R}^{n+1}, \quad f = (f^1, \dots, f^{n+1}),$$

*is defined by eigenfunctions  $f^i$  of the Laplace-Beltrami operator  $\Delta$  with a common eigenvalue  $\lambda$ ,*

$$\Delta f^i = \lambda f^i,$$

*then (i) the image  $f(M)$  lies on the sphere  $\mathbb{S}_R^n$  of radius  $R$  with the center at the origin such that*

$$(2) \quad \lambda = \frac{\dim M}{R^2},$$

*(ii) the immersion  $f : M \looparrowright \mathbb{S}_R^n$  is minimal.*

*If*

$$f : M \looparrowright \mathbb{S}_R^n \subset \mathbb{R}^{n+1}, \quad f = (f^1, \dots, f^{n+1}),$$

*is a minimal isometric immersion of a manifold  $M$  into the sphere  $\mathbb{S}_R^n$  of radius  $R$ , then  $f^i$  are eigenfunctions of the Laplace-Beltrami operator  $\Delta$ ,*

$$\Delta f^i = \lambda f^i,$$

*with the same eigenvalue  $\lambda$  given by formula (2).*

Let us introduce the Weyl eigenvalues counting function

$$N(\lambda) = \#\{\lambda_i | \lambda_i < \lambda\}.$$

The Takahashi Theorem 2.2 implies that if  $M$  is isometrically minimally immersed in the sphere  $\mathbb{S}_R^n$ , then among the eigenvalues of  $M$  there are at least  $n+1$  eigenvalues equal to  $\frac{\dim M}{R^2}$ . It is easy to see that  $\lambda_{N(\frac{\dim M}{R^2})}$  is the first eigenvalue equal to  $\frac{\dim M}{R^2}$ . It is important due to the following theorem.

**Theorem 2.3** (Nadirashvili [32], El Soufi, Ilias [16]). *Let  $M \looparrowright \mathbb{S}_R^n$  be an immersed minimal compact submanifold. Then the metric induced on  $M$  by this immersion is extremal for the functional  $\bar{\lambda}_{N(\frac{\dim M}{R^2})}(M, g)$ .*

*If a metric on a compact manifold  $M$  is extremal then there exists an isometric minimal immersion to the sphere  $M \looparrowright \mathbb{S}_R^n$  by eigenfunctions with eigenvalue  $\frac{\dim M}{R^2}$  of the Laplace-Beltrami operator corresponding to this metric.*

If a metric is extremal for  $\bar{\lambda}_i(M, g)$ , then there exist a minimal immersion of  $M$  by corresponding eigenfunctions into  $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ . If the image is not contained in some hyperplane then one should have at least  $n+1$  linearly independent eigenfunction. This means that  $n+1 \leq m(M, g, \lambda_i)$ .

It follows that if we have an upper bound on the multiplicity of an eigenvalue then we have an upper bound on the dimension of the sphere where  $M$  is minimally immersed by the corresponding eigenfunctions.

We use later Theorem 2.3 for  $M = \mathbb{RP}^2$ . In this case  $\dim M = 2$ . Using rescaling one can consider only the case of  $R = 1$ . Remark that Theorem 2.3 holds also for a non-orientable  $M$ .

Since we are interested in the functional  $\bar{\lambda}_2(\mathbb{RP}^2, g)$ , we need an upper bound for  $m(\mathbb{RP}^2, g, \lambda_2)$  in order to understand the dimension of the sphere which is sufficient to consider.

### 3. NODAL GRAPHS AND COURANT NODAL DOMAIN THEOREM

In this Section we recall basics of the theory of nodal graphs and the Courant Nodal Domain Theorem that we need in order to obtain in Section 4 an upper bound  $m(\mathbb{RP}^2, g, \lambda_2) \leq 6$ .

Let us now recall the following Bers theorem.

**Theorem 3.1** (L. Bers [5]). *Let  $(M, g)$  be a compact 2-dimensional closed Riemannian manifold and  $x_0$  is a point on  $M$ . Then there exist its neighbourhood chart  $U$  with coordinates  $x = (x^1, x^2) \in U \subset \mathbb{R}^2$  centered at  $x_0$  such that for any eigenfunction  $u$  of the Laplace-Beltrami operator on  $M$  there exists an integer  $n \geq 0$  and a non-trivial homogeneous harmonic polynomial  $P_n(x)$  of degree  $n$  on the Euclidean plane  $\mathbb{R}^2$  such that*

$$u(x) = P_n(x) + O(|x|^{n+1}),$$

where  $x \in U$ .

The integer  $n$  from Bers Theorem 3.1 is called an order of zero of an eigenfunction  $u$  at a point  $x_0$ . Let us denote it by  $\text{ord}_{x_0} u$ .

Consider sets

$$N^l(u) = \{x \in M \mid \text{ord}_x u \geq l\}.$$

**Definition 3.2.** *The set  $N^1(u)$  is called a nodal set of  $u$ . Connected components of its complement  $M \setminus N^1(u)$  are called nodal domains of  $u$ .*

It is well-known that in the polar coordinates  $r, \varphi$  in  $\mathbb{R}^2$  any homogeneous harmonic polynomial  $P_n$  of degree  $n$  has the form

$$(3) \quad P_n(r, \varphi) = r^n (A \cos n\varphi + B \sin n\varphi).$$

The zeroes of such polynomials form  $n$  straight lines intersecting at origin at equal angles.

It follows that the nodal set  $N^1(u)$  is a graph such that the points of  $N^2(u)$  are its vertices and the connected components of  $N^1(u) \setminus N^2(u)$  are its edges.

**Definition 3.3.** *This graph is called a nodal graph of an eigenfunction  $u$ .*

Let us remark that if  $x_0$  is a vertex of the nodal graph then it is zero of  $u$  and there is  $2 \operatorname{ord}_{x_0} u$  edges emanating from  $x_0$  in a sufficiently small neighborhood of  $x_0$ . Globally some of these edges could form loops starting and ending at  $x_0$ .

Let us remark that *locally in a neighborhood of zero  $x_0$  of order  $n$  the nodal graph  $N^1(u)$  looks like a star consisting of  $2n$  rays with equal angles between adjacent rays.* Let us give the following definition in order to be more precise.

**Definition 3.4.** *A star  $S_{x_0}(N^1(u))$  at the vertex  $x_0$  of the nodal graph  $N^1(u)$  of an eigenfunction  $u$  consists of  $2n$  unitary tangent vectors to edges emanating from  $x_0$ , where  $n$  is the order of zero of  $u$  at  $x_0$ .*

It follows from formula (3) that in coordinates given by the Bers Theorem 3.1 the angles between adjacent vectors in  $S_{x_0}(N^1(u))$  are equal.

If one has a triangulation of a surface  $M$  with  $V$  vertices,  $E$  edges and  $F$  faces, then one has the well-known formula for the Euler characteristic,

$$(4) \quad \chi(M) = V - E + F.$$

Let us consider an eigenfunction  $u$ . If we consider the vertices of a nodal graph, the edges of a nodal graph and the nodal domains of a function  $u$ , then the formula (4) does not in general hold since the nodal domains are not in general homeomorphic to a disc. As a result, we obtain in this case only the Euler inequality

$$(5) \quad \chi(M) \leq V - E + F$$

that implies the following well-known Lemma.

**Lemma 3.5.** *Let  $u$  be an eigenfunction. Let  $x_j$ ,  $j = 1, \dots, n$ , be zeroes of  $u$  of order  $m_j > 1$ . Let  $\Omega_j$ ,  $j = 1, \dots, s$ , be nodal domains of the function  $u$ . Then*

$$s \geq \chi(M) - n + \sum_{j=1}^n m_j.$$



**Proof.** One see immediately that  $V = n$ ,  $F = s$ . Since  $2 \operatorname{ord}_{x_j} u = 2m_j$  edges emanate from  $x_j$  and each edge is connected to two vertices, one has  $E = \sum_{j=1}^n m_j$ . It is sufficient now to apply inequality (5).  $\square$

Let us now recall the following theorem (remark that we count eigenvalues starting from  $\lambda_0$ ).

**Theorem 3.6** (Courant Nodal Domain Theorem [12]). *An eigenfunction corresponding to the eigenvalue  $\lambda_i$  has at most  $i+1$  nodal domains.*

Lemma 3.5 and Courant Nodal Domain Theorem 3.6 imply immediately the following Proposition.

**Proposition 3.7.** *Let  $u$  be an eigenfunction corresponding to the eigenvalue  $\lambda_i$ . Let  $x_j$ ,  $j = 1, \dots, n$ , be zeroes of  $u$  of order  $m_j > 1$ . Then*

$$(6) \quad i + 1 \geq \chi(M) - n + \sum_{j=1}^n m_j.$$

#### 4. MULTIPLICITY OF THE SECOND NON-ZERO EIGENVALUE OF THE LAPLACE-BELTRAMI OPERATOR ON THE PROJECTIVE PLANE

It was proven by the first author in the paper [31] that the following upper bound for the multiplicities of the eigenvalues of the Laplace-Beltrami operator on the projective plane holds,

$$m(\mathbb{RP}^2, g, \lambda_i) \leq 2i + 3.$$

For the second eigenvalue this means

$$m(\mathbb{RP}^2, g, \lambda_2) \leq 7.$$

The main goal of this Section is to improve this upper bound.

**Proposition 4.1.** *The following upper bound for the multiplicity of the second eigenvalue of the Laplace-Beltrami operator on the projective plane holds,*

$$(7) \quad m(\mathbb{RP}^2, g, \lambda_2) \leq 6.$$

For the purposes of the present paper the upper bound (7) is sufficient. However, this bound is further improved and generalized in the separate paper [3].

Let us postpone the proof and start with several lemmas.

**Lemma 4.2.** *Let  $u_1, \dots, u_6$  be linearly independent eigenfunctions corresponding to the second eigenvalue  $\lambda_2(\mathbb{RP}^2, g)$ . Then for any point  $x_0 \in \mathbb{RP}^2$  there exists a non-trivial linear combination  $v = \sum_{i=1}^6 \alpha_i u_i$  such that the eigenfunction  $v$  has a zero of order at least 3 at the point  $x_0$ .*

**Proof.** This lemma is a particular case of Lemma 4 from paper [31]. In fact, the proof is an easy corollary of Bers Theorem 3.1 and formula (3).  $\square$

**Lemma 4.3.** *Let  $u$  be an eigenfunction corresponding to the second eigenvalue  $\lambda_2(\mathbb{RP}^2, g)$  such that at a point  $x_1$  this eigenfunction has a zero of order at least 3. Then  $x_1$  is the only zero of  $u$  of order greater than 1 and the order of zero at  $x_1$  is exactly 3.*

**Proof.** Since  $i = 2$ ,  $\chi(\mathbb{RP}^2) = 1$ , inequality (6) implies in this case the inequality

$$2 \geq \sum_{j=1}^n (m_j - 1).$$

Since  $m_1 \geq 3$  and  $m_i \geq 2$  for  $i > 1$ , we have  $m_1 - 1 \geq 2$ ,  $m_i - 1 \geq 1$  for  $i > 1$ . It follows that  $m_1 = 3$  and  $n = 1$ .  $\square$

Let us fix a point  $x_0 \in \mathbb{RP}^2$  and consider the space  $V$  of eigenfunctions of  $\Delta$  corresponding to the second eigenvalue  $\lambda_2(\mathbb{RP}^2, g)$  with zero of order at least 3 at  $x_0$ . Let us suppose that  $\dim V \geq 2$ . Then there exist two linearly independent eigenfunctions  $u_1, u_2 \in V$ . Consider the family of eigenfunctions

$$(8) \quad v^\tau = \cos \tau u_1 + \sin \tau u_2.$$

**Lemma 4.4.** *The star  $S_{x_0}(N^1(v^\tau))$  defines the eigenfunction  $v^\tau$  from formula (8) completely up to a sign, i.e. if  $S_{x_0}(N^1(v^{\tau_1})) = S_{x_0}(N^1(v^{\tau_2}))$  then  $v^{\tau_1} = \pm v^{\tau_2}$ .*

**Proof.** Since  $S_{x_0}(N^1(v^{\tau_1})) = S_{x_0}(N^1(v^{\tau_2}))$ , the homogeneous harmonic polynomials  $P_3^{\tau_1}$  and  $P_3^{\tau_2}$  corresponding by Bers theorem 3.1 to  $v^{\tau_1}$  and  $v^{\tau_2}$  are proportional. But then formula (8) implies that either  $P_3^{\tau_1} = P_3^{\tau_2}$  or  $P_3^{\tau_1} = -P_3^{\tau_2}$ . In the first case we have

$$v^{\tau_1} - v^{\tau_2} = O(|x|^4).$$

Then  $v^{\tau_1} - v^{\tau_2}$  is an eigenfunction of  $\Delta$  corresponding to the second eigenvalue  $\lambda_2(\mathbb{RP}^2, g)$  with zero of order at least 4 at  $x_0$ . It follows from Lemma 4.3 that  $v^{\tau_1} - v^{\tau_2} \equiv 0$  and therefore  $v^{\tau_1} = v^{\tau_2}$ . A similar argument shows that in the second case we have  $v^{\tau_1} = -v^{\tau_2}$ .  $\square$

**Lemma 4.5.** *Let  $x_0 \in \mathbb{RP}^2$  and  $V$  be the space of eigenfunctions of  $\Delta$  corresponding to the second eigenvalue  $\lambda_2(\mathbb{RP}^2, g)$  with zero of order at least 3 at  $x_0$ . Then  $\dim V \leq 1$ .*

**Proof.** Let us suppose that  $\dim V \geq 2$ . Then there exist two linearly independent eigenfunctions  $u_1, u_2 \in V$ . Consider the family of

eigenfunctions  $v^\tau \in V$  defined by equation (8) and the family of nodal graphs  $N^1(v^\tau)$ .

Since  $v^0 = -v^\pi$ , the nodal graph  $N^1(v^\tau)$  is deformed continuously when  $\tau$  changes from 0 to  $\pi$  and the result coincides with the initial graph,  $N^1(v^0) = N^1(v^\pi)$ .

Since  $N^1(v^0) = N^1(v^\pi)$ , when  $\tau$  changes from 0 to  $\pi$  the 6-ray star  $S_{x_0}(N^1(v^\tau))$  rotates by angle  $k\frac{\pi}{3}$ . But then  $k = \pm 1$ . Indeed, if  $k \neq \pm 1$  then there exists  $0 < \tau_0 < \pi$  such that  $S_{x_0}(N^1(v^{\tau_0}))$  is obtained from  $S_{x_0}(N^1(v^0))$  by the rotation by angle  $(\operatorname{sgn} k)\frac{\pi}{3}$ . Then  $S_{x_0}(N^1(v^{\tau_0})) = S_{x_0}(N^1(v^0))$  and Lemma 4.4 implies that  $v^{\tau_0} = \pm v^0$ , but this contradicts the inequality  $0 < \tau_0 < \pi$ .

Let us change the direction of counting the angle in such a way that the angle of rotation is  $\frac{\pi}{3}$ . Then we have the following result: *when  $\tau$  changes from 0 to  $\pi$  the star  $S_{x_0}(N^1(v^\tau))$  rotates exactly by  $\frac{\pi}{3}$ .*

Let  $p : \mathbb{S}^2 \rightarrow \mathbb{RP}^2$  be the standard projection. Let us consider the eigenfunction  $u_1 \circ p$  on the sphere  $\mathbb{S}^2$ . It follows from the above mentioned arguments that the nodal graph  $N^1(u_1 \circ p)$  on the sphere has the following properties:

- there are exactly two vertices  $p^{-1}(x_0)$  that we call  $N$  and  $S$ , they are antipodal,
- locally exactly 6 edges emanate from each vertex,
- the nodal graph is invariant with respect to the rotation by  $\frac{\pi}{3}$  around the axis going through  $N$  and  $S$ .

Let us emphasize that “invariant” here and below means “invariant up to a homotopy preserving tangent vectors at the point  $N$  and  $S$ ”.

Let us consider an edge emanating from  $N$ . If its another endpoint is also  $N$  then the nodal graph could not be invariant with respect to the rotation by  $\frac{\pi}{3}$  around the axis going through  $N$  and  $S$ . Indeed, let us numerate the vectors from the star  $S_N(N^1(u_1 \circ p))$  in consecutive order as  $v_0, \dots, v_5$ . Consider the edge  $\gamma$  emanating from  $N$  with tangent vector  $v_0$ . If its tangent vector at the endpoint  $N$  is  $-v_1$  then the nodal graph is clearly not invariant under the rotation by  $\frac{\pi}{3}$ . If its tangent vector at the endpoint  $N$  is  $-v_k$ , where  $k = 2$  or  $k = 3$ , then the edge emanating from  $N$  with tangent vector  $v_1$  has  $-v_{k+1}$  as its tangent vector at its endpoint  $N$ . This implies that there are two loops on a sphere intersecting transversally at exactly one point but this is impossible. The cases  $k = 4$  and  $k = 5$  could be considered as cases  $k = 2$  and  $k = 1$  with another numeration order of the vectors from the  $S_N(N^1(u_1 \circ p))$ .

Hence another endpoint of an edge emanating from  $N$  is  $S$ . Then due to the rotational symmetry all edges go from  $N$  to  $S$ .

Let us introduce spherical coordinates  $0 \leq \varphi < 2\pi$ ,  $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$  such that vectors belonging to the star  $S_N(N^1(u_1 \circ p))$  are tangent to the meridians

$$\begin{cases} \varphi = \frac{\pi}{3}k, \\ \psi = t, \end{cases}$$

where  $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$  and  $k = 0, \dots, 5$ .

Let  $P(t) = \frac{\pi}{6} + \frac{t}{2} - \frac{2t^3}{3\pi^2}$ . This polynomial is monotone on  $(-\frac{\pi}{2}, \frac{\pi}{2})$  and the following properties hold,  $P(-\frac{\pi}{2}) = 0$ ,  $P'(-\frac{\pi}{2}) = 0$ ,  $P(+\frac{\pi}{2}) = \frac{\pi}{3}$ ,  $P'(\frac{\pi}{2}) = 0$ .

All edges start at  $N$  and end at  $S$  and do not intersect except at the only vertices  $N$  and  $S$ . It follows that there exists an integer  $n$  such that all edges are homotopic to paths

$$\begin{cases} \varphi = \frac{\pi}{3}k + nP(t), \\ \psi = t, \end{cases}$$

where  $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ ,  $k = 0, \dots, 5$ , and the homotopy is such that a) the edges are not intersecting, b) the tangent vectors at  $N$  and  $S$  are preserved.

One can easily check that these paths are invariant under the antipodal map only for  $n = 0$ . This means that there are exactly 6 nodal domains looking (up to a homotopy preserving stars at  $N$  and  $S$ ) as orange segments.

Let us now recall that signs of an eigenfunction in adjacent nodal domains are opposite. This implies that the signs of the eigenfunction  $u_1 \circ p$  in antipodal nodal domains are different, but then this eigenfunction on the sphere cannot be an eigenfunction lifted from the projective plane. This contradiction means that the initial assumption  $\dim V \geq 2$  was wrong.  $\square$

**Proof** of Proposition 4.1. Let us suppose that  $m(\mathbb{RP}^2, g, \lambda_2) > 6$ . Then there exist 7 linearly independent eigenfunctions  $\varphi_1, \dots, \varphi_7$  corresponding to the second eigenvalue  $\lambda_2(\mathbb{RP}^2, g)$ .

Let us fix a point  $x_0 \in \mathbb{RP}^2$ . Let us apply Lemma 4.2 to  $\varphi_1, \dots, \varphi_6$  and obtain an eigenfunction  $u_1 = \sum_{i=1}^6 \alpha_i \varphi_i$  with zero of order at least 3 at the point  $x_0$ . Then by Lemma 4.3 the point  $x_0$  is a zero of order exactly 3.

We can suppose without loss of generality that  $\alpha_1 \neq 0$ . Let us then apply Lemma 4.2 to the eigenfunctions  $\varphi_2, \dots, \varphi_7$  and obtain an eigenfunction  $u_2 = \sum_{i=2}^7 \beta_i \varphi_i$  with zero of order at least 3 at the point  $x_0$ . Then by Lemma 4.3 the point  $x_0$  is a zero of order exactly 3.

Let us remark that  $u_1$  and  $u_2$  are linearly independent since  $\alpha_1 \neq 0$ . This contradicts Lemma 4.5.  $\square$

## 5. HARMONIC MAPS WITH BRANCH POINTS AND METRICS WITH CONICAL SINGULARITIES

Let us recall the definition of a harmonic map, see e.g. the review [13].

**Definition 5.1.** *Let  $(M, g)$  and  $(N, h)$  be Riemannian manifolds. A smooth map  $f : M \rightarrow N$  is called harmonic if  $f$  is an extremal for the energy functional*

$$(9) \quad E[f] = \int_M |df(x)|^2 dVol_g.$$

The following theorem (see, e.g. the paper [14]) explains the relation between minimal and harmonic maps in the class of isometric immersions.

**Theorem 5.2.** *Let  $M, N$  be Riemannian manifolds. If  $f : M \rightarrow N$  is an isometric immersion, then  $f$  is harmonic if and only if  $f$  is minimal.*

Theorem 2.3 and Theorem 5.2 imply the following Proposition.

**Proposition 5.3.** *The extremal metrics on a compact surface  $M$  are exactly the metrics induced on  $M$  by harmonic immersions  $M \rightarrow \mathbb{S}^n$ .*

It turns out however that it is useful to consider a wider class of harmonic immersions with branch points.

**Definition 5.4** (see e.g. [18]). *Let  $M$  be a manifold of dimension 2. A smooth map  $f : M \rightarrow N$  has a branch point of order  $k$  at point  $p$  if there exist local coordinates  $u_1, u_2$  centered at  $p$  and defined in a neighborhood of  $p$  and  $x_1, \dots, x_n$  centered at  $f(p)$  and defined in a neighborhood of  $f(p)$  such that in these coordinates  $f$  is written as*

$$\begin{aligned} x_1 + ix_2 &= w^{k+1} + \sigma(w), \\ x_k &= \chi_k(w), \quad k = 3, \dots, n, \\ \sigma(w), \chi_k(w) &= o(|w|^{k+1}), \\ \frac{\partial \sigma}{\partial u_j}(w), \frac{\partial \chi_k}{\partial u_j}(w) &= o(|w|^k), \quad j = 1, 2, \quad k = 3, \dots, n, \end{aligned}$$

where  $w = u^1 + iu^2$ .

If  $M$  is compact then a map  $f : M \rightarrow N$  could have only finite number of branch points.

However we have now a problem. If  $(N, g)$  is a Riemannian manifold and  $f : M \looparrowright N$  is an immersion with branch points, then the induced metric  $f^*g$  is not a smooth metric.

**Definition 5.5** (see e.g. [26]). *A point  $p$  on a surface  $M$  is called a conical singularity of order  $\alpha > -1$  or angle  $2\pi(\alpha + 1)$  of a metric  $g$  if in an appropriate local complex coordinate  $z$  centered at  $p$  the metric has the form*

$$g(z) = |z|^{2\alpha} \rho(z) |dz|^2$$

*in a neighborhood of  $p$ , where  $\rho(0) > 0$ .*

Then we obtain immediately the following Proposition.

**Proposition 5.6.** *If  $M$  is a compact surface,  $(N, h)$  is a Riemannian manifold and  $f : M \looparrowright N$  is an immersion with branch point, then  $g = f^*h$  is a smooth Riemannian metric except a finite number of branch points of the map  $f$ . At these points the metric  $g$  has conical singularities. The order of the conical singularity at a point  $p$  is equal to the order of  $p$  as a branch point.*

Thus, we switch to a setting larger than the initial one. We consider not only Riemannian metrics but also Riemannian metrics with a finite number of conical singularities and not only harmonic immersions but also harmonic immersions with branch points. Then we should check that all key results from the previous sections hold.

It is well-known that the eigenvalues of the Laplace-Beltrami operator could be defined using a variational approach,

$$(10) \quad \lambda_k = \min_{\substack{V \subset H^1(M) \\ \dim V = k}} \max_{\substack{u \in V \\ u \perp 1}} R[v],$$

where

$$R[v] = \frac{\int_M |\nabla u|^2 dVol}{\int_M |u|^2 dVol}$$

is the Rayleigh quotient. This formula holds also in the case of metrics with conical singularities, see e.g. [26].

Branch points does not affect convergence neither of the energy functional (9) nor the integrals in the proof of Theorem 2.3. Hence Proposition 5.3 holds also in the setting of metrics with conical singularities and harmonic maps with branch points.

We could have problems with Section 3. A priori zeroes of eigenfunctions could accumulate to conical singularities. If this is true then  $V$ ,  $E$  and  $F$  are infinite and inequality (5) does not hold. However, it is not true. In the particular case of  $\mathbb{RP}^2$  the metrics induced by harmonic maps are analytic due to Morrey's regularity results, see the

book [30]. This implies that accumulation of zeroes is impossible, and the nodal graph is finite. Let us remark that in a general situation one could use an approach similar to the approach used in the paper [25, Lemma 3.1.1] in order to prove the finiteness of a nodal graph. Thus, in the setting of metrics with conical singularities inequality (5) and all results obtained with its help hold, including the key upper bound  $m(\mathbb{RP}^2, g, \lambda_2) \leq 6$  from Proposition 4.1 from Section 4.

Let us also remark that for any manifold equipped with a metric with isolated conical singularities it is possible to construct a sequence of smooth Riemannian manifolds such that their area as well as their eigenvalues converge to the area and eigenvalues of the initial manifold, see e.g. [43].

## 6. CALABI-BARBOSA THEOREM AND ITS IMPLICATIONS

Now we should study harmonic immersions with branched points  $\mathbb{RP}^2 \looparrowright \mathbb{S}^n$ . Since we have the upper bound  $m(\mathbb{RP}^2, g, \lambda_2) \leq 6$  from Proposition 4.1, all immersions corresponding to  $\lambda_2$  are among immersions  $\mathbb{RP}^2 \looparrowright \mathbb{S}^5$ .

Let  $p : \mathbb{S}^2 \rightarrow \mathbb{RP}^2$  be the standard projection. We can lift a harmonic immersion with branch points  $f : \mathbb{RP}^2 \looparrowright \mathbb{S}^5$  to a harmonic immersion with branch points  $F = f \circ p : \mathbb{S}^2 \rightarrow \mathbb{S}^5$ .

The following theorem was proved by Calabi in 1967 and later refined by Barbosa in 1975. Let  $g_{\mathbb{S}^n}$  denote the standard metric on  $\mathbb{S}^n$ . The radius of  $\mathbb{S}^n$  is 1.

**Theorem 6.1** (Calabi [10], Barbosa [2]). *Let  $F : \mathbb{S}^2 \rightarrow \mathbb{S}^n$  be a harmonic immersion with branch points such that the image is not contained in a hyperplane. Then*

- (i) *the area of  $\mathbb{S}^2$  with respect to the induced metric  $\text{Area}(\mathbb{S}^2, F^*g_{\mathbb{S}^n})$  is an integer multiple of  $4\pi$ ;*
- (ii)  *$n$  is even,  $n = 2m$ , and*

$$\text{Area}(\mathbb{S}^2, F^*g_{\mathbb{S}^n}) \geq 2\pi m(m+1).$$

**Definition 6.2.** *If  $\text{Area}(\mathbb{S}^2, F^*g_{\mathbb{S}^n}) = 4\pi d$ , then we say that  $F$  is of harmonic degree  $d$ .*

We obtain immediately a lower bound for the harmonic degree.

**Proposition 6.3.** *Let  $F : \mathbb{S}^2 \rightarrow \mathbb{S}^{2m}$  be a harmonic immersion with branch points such that the image is not contained in a hyperplane. Then  $d \geq \frac{m(m+1)}{2}$ .*

Calabi-Barbosa Theorem 6.1 implies the following Proposition.

**Proposition 6.4.** *It is sufficient for our goals to consider harmonic immersions with branch points  $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$  (such that the image is not contained in a hyperplane) of harmonic degree  $d \geq 3$  and  $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ .*

It follows that we should consider only harmonic immersions with branch points  $\mathbb{RP}^2 \longrightarrow \mathbb{S}^2$  and  $\mathbb{RP}^2 \longrightarrow \mathbb{S}^4$ . However, the following Proposition permits to exclude maps  $\mathbb{RP}^2 \longrightarrow \mathbb{S}^2$ .

**Proposition 6.5** (see e.g. [13]). *Every harmonic map  $\mathbb{RP}^2 \longrightarrow \mathbb{S}^2$  is constant.*

## 7. HARMONIC MAPS FROM $\mathbb{S}^2$ TO $\mathbb{S}^4$ AND THEIR SINGULARITIES

Let us recall the well-known Penrose twistor map

$$T : \mathbb{CP}^3 \longrightarrow \mathbb{HP}^1 \cong \mathbb{S}^4, \quad T([z_0 : z_1 : z_2 : z_3]) = [z_0 + z_1 j : z_2 + z_3 j].$$

Let  $z$  be a conformal parameter on  $\mathbb{S}^2$ .

**Definition 7.1.** *Let us call a curve*

$$f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3, \quad f(z) = [f_0(z) : f_1(z) : f_2(z) : f_3(z)],$$

*horizontal if*

$$f'_1 f_2 - f_1 f'_2 + f'_3 f_4 - f_3 f'_4 = 0.$$

In 1982 Bryant described in the paper [9] a very important relation between harmonic immersions with branched points  $\mathbb{S}^2 \longrightarrow \mathbb{S}^4$  and (anti)holomorphic horizontal curves in  $\mathbb{CP}^3$ .

Let  $A : \mathbb{S}^4 \longrightarrow \mathbb{S}^4$  be the antipodal map.

**Theorem 7.2** (Bryant [9]). *For each harmonic immersion with branch points  $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$  there exist either holomorphic or antiholomorphic horizontal curve  $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$ , such that  $T \circ f = F$ ,*

$$\begin{array}{ccc} & & \mathbb{CP}^3 \\ & \nearrow f & \downarrow T \\ \mathbb{S}^2 & \xrightarrow{F} & \mathbb{S}^4 \end{array}$$

*For each (anti)holomorphic horizontal curve  $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$  the map  $F = T \circ f : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$  is a harmonic immersion with branched points.*

*If a harmonic immersion  $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$  has a holomorphic (anti-holomorphic) horizontal curve  $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$ , then  $A \circ F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$  has an antiholomorphic (holomorphic) horizontal curve.*

**Definition 7.3.** *An (anti)holomorphic horizontal curve  $f$  appearing in Bryant Theorem 7.2 is called the lift of an harmonic immersion  $F$ .*



Let us remark that  $F$  and  $A \circ F$  induce the same metric on  $\mathbb{S}^2$ . It follows that it is sufficient to consider harmonic immersions with holomorphic lifts.

**Theorem 7.4** (Bryant [9]). *Let  $F : \mathbb{S}^2 \rightarrow \mathbb{S}^4$  be a harmonic immersion with branched points of harmonic degree  $d$  with holomorphic lift  $f : \mathbb{S}^2 \rightarrow \mathbb{CP}^3$ . Then  $f : \mathbb{S}^2 \rightarrow \mathbb{CP}^3$  is an algebraic curve of degree  $d$ .*

Now we need some results from the theory of higher singularities of these holomorphic horizontal lifts, see e.g. the paper [7] by Bolton and Woodward. It is impossible to expose this theory here in details, let us present a very short summary of results necessary for our goals. A notion of a higher singularity of a holomorphic curve is defined through singularities of its Frenet frame. A higher singularity at a point  $p$  is described by its singularity type  $(p, r_0(p), r_1(p), r_2(p))$ , where  $r_i$  are non-negative integers. For a horizontal curve  $r_2(p) = r_0(p)$ , i.e. its higher singularity type at a point  $p$  is described by two integers  $r_0(p)$ , and  $r_1(p)$ . Let us define quantities

$$r_0 = \sum_p r_0(p), \quad r_1 = \sum_p r_1(p).$$

The next Proposition relates them to the degree  $d$ .

**Proposition 7.5** (Bolton, Woodward [8]). *For a linearly full holomorphic horizontal curve in  $\mathbb{CP}^3$  the following equation holds,*

$$2r_0 + r_1 = 2d - 6.$$

We need here to recall the definition of an umbilic point.

**Definition 7.6.** *Let  $(M, g)$  and  $(N, h)$  be Riemannian manifolds. Let  $F : M \rightarrow N$  be an immersion. Then the second fundamental form  $\mathbf{II}^F$  of  $F$  is defined by the formula*

$$\nabla_{dF(X)}^N dF(Y) = dF(\nabla_X^M Y) + \mathbf{II}^F(X, Y).$$

*The vector field*

$$\zeta = \frac{1}{\dim M} \operatorname{tr} \mathbf{II}^F$$

*is called a mean curvature normal vector.*

*A point  $p \in M$  is called an umbilic if there exists a vector  $v \in T_{F(p)}N$  such that at the point  $p$  one has*

$$\mathbf{II}_p^F(X, Y) = g_p(X, Y) \cdot v.$$

It follows immediately from Definition 7.6 that if  $p$  is an umbilic then  $\mathbf{II}_p^F(X, Y) = g_p(X, Y) \cdot \zeta(p)$ .

In our situation of a harmonic immersion  $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$  it is convenient to use a conformal parameter  $z$  on  $\mathbb{S}^2$ . It is easy to check that the following Proposition holds.

**Proposition 7.7.** *A point  $p \in \mathbb{S}^2$  is an umbilic of a harmonic immersion  $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$  if and only if*

$$\mathbf{II}_p^F(\partial/\partial z, \partial/\partial z) = 0.$$

The higher singularities of a holomorphic horizontal lift  $f$  of a harmonic immersion with branched points  $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$  are related to the branch points and the umbilics of  $F$ .

**Proposition 7.8** (Bolton, Woodward [7, 8]). *A point  $p$  is a branch point of  $F$  if and only if  $r_0(p) > 0$ . Moreover,  $r_0(p)$  is equal to the order of zero of  $dF(\partial/\partial z)$  at  $p$ .*

*If  $r_0(p) = 0$  then  $p$  is an umbilic if and only if  $r_1(p) > 0$ . Moreover,  $r_1(p)$  is equal to the order of zero of  $\mathbf{II}^F(\partial/\partial z, \partial/\partial z)$  at  $p$ .*

*The higher singularities of  $f$  occur exactly at the branch points and umbilics of  $F$ .*

Combining Propositions 7.5 and 7.8, we obtain the following Proposition.

**Proposition 7.9.** *Let  $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$  be a harmonic immersion with branch points of harmonic degree  $d$ . Then*

- (i) *if  $d = 3$  then  $F$  does not have neither branch points nor umbilics,*
- (ii) *if  $d > 3$  then  $F$  has at least one branch point or umbilic.*

## 8. MAXIMIZATION OF HIGHER ORDER EIGENVALUES IN A CONFORMAL CLASS AND THE BUBBLING PHENOMENON

The last ingredient of the proof of main Theorem 1.3 is the theory of maximization of higher eigenvalues in a conformal class developed by the first author and Sire in the papers [34, 35].

Consider a Riemannian metric  $g$  on a connected compact closed surface  $M$ . Let us denote by  $[g]$  the following class of metrics conformally equivalent to  $g$ ,

$$[g] = \{\tilde{g} | \tilde{g} = \mu g\},$$

where  $\mu : M \rightarrow \mathbb{R}^+$  is an  $L^1$  function on  $M$  with mass 1, i.e. a probability density. Usually  $[g]$  is called the conformal class of  $g$ . Remark, however, that  $[g]$  contains only metrics of area 1 conformally equivalent to  $g$ .

Let us consider the supremum of an eigenvalue over metrics of area 1 conformally equivalent to  $g$ ,

$$\tilde{\Lambda}_k(M, [g]) = \sup_{\tilde{g} \in [g]} \lambda_k(\tilde{g}).$$

Let  $g_{\text{round}}$  denote the standard metric on the sphere  $\mathbb{S}^2$  of radius 1.

**Theorem 8.1** (Nadirashvili, Sire [35]). *Let  $(M, g)$  be a smooth connected compact boundaryless Riemannian surface. For any  $k \geq 1$  and a sequence of metrics  $\{g'_i\}_{i \geq 1} \in [g]$  of the form  $g'_i = \mu'_i g$  such that*

$$\lim_{i \rightarrow \infty} \lambda_k(g'_i) = \tilde{\Lambda}_k(M, [g])$$

*there exists a subsequence of metrics  $\{g_n\}_{n \geq 1} = \{g'_{i_n}\}_{n \geq 1} \in [g]$ , where  $g_n = \mu_n g$ , such that*

$$\lim_{n \rightarrow \infty} \lambda_k(g_n) = \tilde{\Lambda}_k(M, [g])$$

*and a probability measure  $\mu$  such that*

$$\mu_n \rightharpoonup^* \mu \text{ weakly in measure as } n \rightarrow +\infty.$$

*Moreover, the following decomposition holds,*

$$\mu = \mu_r + \mu_s$$

*where  $\mu_r$  is a nonnegative  $C^\infty$  function and  $\mu_s$  is the singular part given, if not trivial, by the formula*

$$\mu_s = \sum_{i=1}^K c_i \delta_{x_i}$$

*for some  $K \geq 1$ ,  $c_i \geq 0$  and some “bubbling points”  $x_i \in M$ . Furthermore, the number  $K$  satisfies the bound*

$$K \leq k - 1.$$

*Moreover, there exist  $m_j$  such that  $1 \leq m_j \leq k$  and*

$$c_j = \frac{\tilde{\Lambda}_{m_j}(\mathbb{S}^2, [g_{\text{round}}])}{\tilde{\Lambda}_k(M, [g])}.$$

*The regular part of the limit density  $\mu$ , i.e.  $\mu_r$ , is either identically zero or  $\mu_r$  is absolutely continuous with respect to the Riemannian measure with a smooth positive density vanishing at most at a finite number of points on  $M$ .*

Furthermore, if we denote by  $A_r$  the volume of the regular part  $\mu_r$ , i.e.  $A_r = \text{Area}(M, \mu_r g)$ , then either  $A_r = 0$  or there exists  $m_0$  such that  $1 \leq m_0 \leq k$  and

$$A_r = \frac{\tilde{\Lambda}_{m_0}(M, [g])}{\tilde{\Lambda}_k(M, [g])}.$$

Finally, if we denote by  $U$  the eigenspace of the Laplace-Beltrami operator on  $(M, \mu_r g)$  associated to the eigenvalue  $\tilde{\Lambda}_k(M, [g])$ , then there exists a family of eigenvectors  $\{u_1, \dots, u_l\} \subset U$  such that the map

$$\varphi = (u_1, \dots, u_l) : M \rightarrow \mathbb{R}^l$$

is a harmonic isometric immersion into the sphere  $\mathbb{S}^{l-1}$ .

Remark that the conformal class  $[g]$  on  $M$  can be always represented by a real-analytic Riemannian manifold  $(M, g)$ . One can choose for example standard metrics on the sphere and the projective plane, a flat metric on the torus and a constant curvature metric  $g$ , representing  $(M, g)$  as a factorization of the hyperbolic plane, in the case of a surface of genus greater than one. Since  $\varphi$  is a solution of an elliptic system with real-analytic coefficients, it follows that  $\varphi$  is a real-analytic map from  $(M, g)$  to  $\mathbb{S}^{l-1}$ , see the book [30]. In particular, all the eigenfunctions  $u_1, \dots, u_l$  are real analytic functions.

Theorem 8.1 could be interpreted in the following way: the supremum of  $\lambda_k$  is attained as a limit on a sequence of metrics converging to a singular metric on  $M$  with  $K$  spheres touching  $M$  in the points  $x_1, \dots, x_K$ . The restriction of this limit singular metric on  $M$  is  $\mu_r g$  and on the sphere touching  $M$  at  $x_i$  is the metric maximizing  $\bar{\lambda}_{m_j}$  on the sphere of area  $c_i$ . That's why this phenomenon is called "bubbling", these spheres bubbles up out of the surface  $M$ . The metric  $\mu_r g$  has area  $A_r$ . Remark that

$$(11) \quad \sum_{i=1}^K c_i + A_r = 1$$

since the area of  $\mu g$  is equal to 1.

Theorem 8.1 explains that this bubbling phenomenon could occur for  $\tilde{\Lambda}_k$  with  $k > 2$ . Note that in the case  $k = 1$  no bubbling phenomenon occurs, i.e.  $\mu_s$  is always identically zero. For  $k = 2$  and  $k = 3$  the bubbling phenomenon was observed on the example of the sphere in the papers [33, 36].

Let us explain the mechanism of bubbling in more details. If in Theorem 8.1 we have  $\mu_s \neq 0$  then we say that the sequence of metrics  $g_n$  is bubbling at points  $x_i$  and  $c_i$  is a part of total area which is concentrated

at the point  $x_i$ . One can choose a sequence of disks  $\mathbb{D}_k$  on  $(M, g)$  centered at  $x_i$  with radii tending to zero and a subsequence  $g_{n_k}$  such that disks  $(\mathbb{D}_k, g_{n_k})$  as Riemannian manifolds are tending to  $(\mathbb{D}, g)$ , where  $g = g_r + g_s$ , with a regular metric  $g_r$  and possibly a nonzero singular part  $g_s$ . The following Lemma 8.2 shows that the spectrum of  $(\mathbb{D}, g)$  is a subset of the limit of spectrums of  $(M, g_n)$ . This Lemma was in fact proved in the paper [35] although it was not stated there in the form presented here.

Let us denote by  $\mathbb{D}_r(x)$  the disk of center  $x \in M$  and of radius  $r$  with respect to the metric  $g$ . The following Lemma describes the decomposition of the spectrum in the case on a singular extremal metric (i.e.  $\mu_s$  is not identically zero), see [36].

**Lemma 8.2.** *There exists a subsequence  $\mu_n g$  of the sequence  $\mu'_i g$  of metrics from Theorem 8.1 with eigenvalues  $\{\lambda_j^n\}_{j \geq 0}$  such that the following property holds.*

*Consider a smooth cut-off function  $\psi_r$  on  $M$  such that  $0 \leq \psi_r \leq 1$ ,  $\psi_r = 0$  on  $\mathbb{D}_{r/2}(\tilde{x})$  and  $\psi_r = 1$  on  $M \setminus \mathbb{D}_r(\tilde{x})$  where  $\tilde{x}$  is a bubbling point in Theorem 8.1. Define the sequence of metrics  $h_n = \psi_{2^{-n}} \mu_n g$  on  $M$  and  $\rho_n$  on  $\mathbb{S}^2$  such that  $(\mathbb{S}^2_+, \rho_n)$  is isometric to  $(\mathbb{D}_{2^{-n}}, \mu_n g - h_n)$ . Let us extend  $\rho_n$  by 0 on  $\mathbb{S}^2_-$ . Denote by  $\{\alpha_i^n\}_{i \geq 0}$  and  $\{\beta_i^n\}_{i \geq 0}$  the sequences of eigenvalues of the Laplace-Beltrami operator on  $(M, h_n)$  and  $(\mathbb{S}^2, \rho_n)$  respectively.*

*Fix a natural number  $N \geq 1$ . Suppose that the following limits exist,*

$$\lim_{n \rightarrow \infty} \lambda_i^n = \lambda_i, \quad i = 0, \dots, N.$$

*Then there exists a subsequence  $n_m$  and natural numbers  $N_1, N_2 \geq 1$  such that the following limits hold,*

$$\lim_{m \rightarrow \infty} \alpha_i^{n_m} = \alpha_i$$

*and*

$$\lim_{m \rightarrow \infty} \beta_i^{n_m} = \beta_i$$

*and, furthermore,*

$$\{\lambda_0, \dots, \lambda_N\} = \{\alpha_0, \dots, \alpha_{N_1}\} \cup \{\beta_0, \dots, \beta_{N_2}\},$$

*where the union of sets is taken considering the multiplicity of the eigenvalues.*

**Remark 8.3.** *Metrics  $\rho_n$  are vanishing on open subsets of  $\mathbb{S}^2$ , however for nonnegative metrics eigenvalues of the Laplace-Beltrami operator are defined via Rayleigh variational formula (10).*

## 9. PROOF OF THEOREM 1.3

Let  $\{g_n\}$  be a maximizing sequence of metrics of area 1 for the functional  $\bar{\lambda}_2(\mathbb{RP}^2, g)$ , i.e.

$$\lim_{n \rightarrow \infty} \bar{\lambda}_2(\mathbb{RP}^2, g_n) = \Lambda_2(\mathbb{RP}^2).$$

Since there is only one conformal structure on  $\mathbb{RP}^2$ , see e.g. the book [42], the metrics  $g_n$  can be written as  $g_n = \mu_n g_{can}$ , where  $g_{can}$  is the canonical metric of area 1 on  $\mathbb{RP}^2$ . It follows that the sequence  $\{g_n\}$  is a maximizing sequence for  $\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{can}])$ . Choosing a subsequence if needed, we can assume that  $\{g_n\}$  is the sequence from Theorem 8.1. Let us use notation from Theorem 8.1 in the rest of the proof.

Let  $\mu_r$  and  $\mu_s$  be the regular part and the singular part from Theorem 8.1. Let us consider three possible cases.

**Case I.** Let  $\mu_s \neq 0$  and  $\mu_r \neq 0$ . Since  $1 \leq K \leq k-1$  and  $k=2$ , one has  $K=1$ , i.e. there is only one bubbling point  $x_1$ .

If  $m_0 = 2$  then  $A_r = 1$  and hence  $c_1 = 0$ . This contradicts the assumption  $\mu_s \neq 0$ . It follows that  $m_0 = 1$ . This implies that

$$(12) \quad A_r = \frac{\tilde{\Lambda}_1(\mathbb{RP}^2, [g_{can}])}{\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{can}])} = \frac{\Lambda_1(\mathbb{RP}^2)}{\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{can}])} = \frac{12\pi}{\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{can}])}$$

due to Li-Yau results from the paper [29].

Let us suppose that  $m_1 = 2$ . Then the metric on the bubbling sphere is maximal for the second eigenvalue on the sphere. It is known from the paper [33] that this is the singular metric on two touching spheres of equal radius with standard metrics. This implies that in Lemma 8.2 one has  $\alpha_0 = \beta_0 = \beta_1 = 0$  and hence  $\lambda_2 = 0$ . Therefore it is not the maximizing sequence. It follows that  $m_1 = 1$  and the metric on the bubbling sphere is maximal for the first eigenvalue on the sphere. It follows from the paper [19] by Hersch that

$$\tilde{\Lambda}_1(\mathbb{S}^2, [g_{ground}]) = \Lambda_1(\mathbb{S}^2) = 8\pi$$

and Theorem 8.1 implies that

$$(13) \quad c_1 = \frac{\tilde{\Lambda}_1(\mathbb{S}^2, [g_{ground}])}{\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{can}])} = \frac{8\pi}{\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{can}])}.$$

The equations (11), (12) and (13) imply

$$(14) \quad \tilde{\Lambda}_2(\mathbb{RP}^2, [g_{can}]) = 20\pi.$$

This result is realized for the canonical metric on  $\mathbb{RP}^2$  with area

$$A_r = \frac{12\pi}{20\pi} = \frac{3}{5}$$

and canonical metric on  $\mathbb{S}^2$  with area

$$c_1 = \frac{8\pi}{20\pi} = \frac{2}{5}.$$

Indeed, for these metrics  $\alpha_0 = \beta_0 = 0$ ,  $\alpha_1 = \beta_1 = 20\pi$ , and Lemma 8.2 implies  $\lambda_2 = 20\pi$ .

**Case II.** Let  $\mu_s \neq 0$  and  $\mu_r = 0$ . As in case I, one has  $K = 1$ , i.e. there is only one bubbling point  $x_1$ . But in this case  $A_r = 0$ . If  $m_1 = 1$  then the metric on the bubbling sphere is maximal for the first eigenvalue on the sphere and equation (11) implies

$$1 = c_1 = \frac{\tilde{\Lambda}_1(\mathbb{S}^2, [g_{\text{round}}])}{\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{\text{can}}])} = \frac{8\pi}{\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{\text{can}}])}.$$

It follows that

$$\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{\text{can}}]) = 8\pi.$$

But this is less than  $20\pi$  in formula (14).

In the same way, if  $m_1 = 2$  then the metric on the bubbling sphere is maximal for the second eigenvalue on the sphere and equation (11) implies

$$1 = c_1 = \frac{\tilde{\Lambda}_2(\mathbb{S}^2, [g_{\text{round}}])}{\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{\text{can}}])} = \frac{16\pi}{\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{\text{can}}])}.$$

It follows that

$$\tilde{\Lambda}_2(\mathbb{RP}^2, [g_{\text{can}}]) = 16\pi.$$

But this is less than  $20\pi$  in formula (14).

**Case III.** Let  $\mu_s = 0$ . In this case Theorem 8.1 implies that  $\mu_r$  is a real-analytic function on  $\mathbb{RP}^2$ .

Let us consider Riemannian metrics on  $\mathbb{RP}^2$  (possibly with conical singularities) extremal for the functional  $\Lambda_2(\mathbb{RP}^2, g)$ . As we already know from Proposition 5.3, these metrics are induced on  $\mathbb{RP}^2$  from harmonic immersions with branched points  $\mathbb{RP}^2 \rightarrow \mathbb{S}^n$ .

The upper bound  $m(\mathbb{RP}^2, g, \lambda_2) \leq 6$  from Proposition 4.1 implies that all harmonic immersions with branch points corresponding to  $\lambda_2$  are among immersions  $\mathbb{RP}^2 \looparrowright \mathbb{S}^5$ .

Let  $p : \mathbb{S}^2 \rightarrow \mathbb{RP}^2$  be the standard projection. We can lift a harmonic immersion with branch points  $f : \mathbb{RP}^2 \looparrowright \mathbb{S}^5$  to a harmonic immersion with branch points  $F = f \circ p : \mathbb{S}^2 \rightarrow \mathbb{S}^5$ .

Calabi-Barbosa Theorem 6.1 implies that it is sufficient to consider harmonic immersions with branch point  $F : \mathbb{S}^2 \rightarrow \mathbb{S}^4$  of harmonic degree  $d \geq 3$  such that the image is not contained in a hyperplane and  $F : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ , see Proposition 6.4. However, Proposition 6.5 says that we can exclude harmonic maps  $\mathbb{RP}^2 \rightarrow \mathbb{S}^2$  since they are constant.

As a result, we should consider only harmonic immersions with branch point  $\mathbb{RP}^2 \longrightarrow \mathbb{S}^4$ .

Consider a harmonic immersions with branch points  $f : \mathbb{RP}^2 \longrightarrow \mathbb{S}^4$  corresponding to  $\lambda_2$  and its lift  $F = f \circ p : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ . As we know from Proposition 7.9, there are two different cases depending on its harmonic degree  $d$ .

Consider the case  $d = 3$ . Let  $g_{\mathbb{S}^n}$  denote the standard metric on  $\mathbb{S}^n$ . Since  $d = 3$ , one has  $\text{Area}(\mathbb{S}^2, F^*g_{\mathbb{S}^n}) = 12\pi$  due to Calabi-Barbosa Theorem 6.1. Then  $\text{Area}(\mathbb{RP}^2, f^*g_{\mathbb{S}^n}) = 6\pi$  because  $p : \mathbb{S}^2 \longrightarrow \mathbb{RP}^2$  is a two-sheeted covering. Since the radius of  $\mathbb{S}^n$  is 1, Takahashi Theorem 2.2 implies that  $\lambda_2 = 2$ . As a result,  $\bar{\lambda}_2(\mathbb{RP}^2, f^*g_{\mathbb{S}^n}) = 12\pi < 20\pi$  and the induced metric is not maximal.

Consider the case  $d > 3$ . In this case Proposition 7.5 implies that  $F = f \circ p : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$  and hence  $f : \mathbb{RP}^2 \longrightarrow \mathbb{S}^4$  have at least one branch point or umbilic. Let us prove that an immersion by eigenfunctions corresponding to  $\lambda_2$  cannot have neither branch points nor umbilics.

Let us suppose that  $f = (f^1, \dots, f^5)$  and  $p \in \mathbb{RP}^2$  is a branch point. It follows that  $f^i$  are linearly independent eigenfunctions with eigenvalue  $\lambda_2 = 2$  such that  $df^i(p) = 0$ . One can then construct at least 4 linearly independent eigenfunctions  $\tilde{f}^i$ ,  $i = 1, \dots, 4$ , such that  $\tilde{f}^i(p) = 0$ ,  $d\tilde{f}^i(p) = 0$ . This means that all  $\tilde{f}^i$  have zero of order 2 at  $p$ . Using Bers Theorem 3.1 one can then construct at least 2 linearly independent eigenfunctions with eigenvalue  $\lambda_2 = 2$  with zero of order 3 at  $p$ , but this contradicts Lemma 4.5.

Let us suppose that  $f = (f^1, \dots, f^5)$  and  $p \in \mathbb{RP}^2$  is an umbilic. Let  $z$  be a local conformal parameter on  $\mathbb{RP}^2$  in a neighborhood of the point  $p$ . Let  $ds^2 = 2\Phi|dz|^2$  be the induced metric. It is well-known that  $f_{z\bar{z}} = -\Phi f$ , see e.g. [2, 10]. Since  $p$  is an umbilic,  $\mathbf{II}_p^f(\partial/\partial z, \partial/\partial z) = 0$ . This means that  $f_{zz}(p)$  is a linear combination of  $f_z(p)$  and  $f_{\bar{z}}(p)$ . It follows that there exist  $\alpha, \beta \in \mathbb{C}$  such that for any  $i = 1, \dots, 5$  the following equations hold,

$$(15) \quad f_{z\bar{z}}^i(p) = -\Phi(p)f^i(p),$$

$$(16) \quad f_{zz}^i(p) = \alpha f_z^i(p) + \beta f_{\bar{z}}^i(p),$$

$$(17) \quad f_{\bar{z}\bar{z}}^i(p) = \bar{\beta} f_z^i(p) + \bar{\alpha} f_{\bar{z}}^i(p).$$

Remark that these equations are linear. This implies that they hold for any linear combination of  $f^i$ .

Now one can construct two linear combinations

$$\varphi = \sum_{i=1}^5 A_i f^i, \quad \psi = \sum_{i=1}^5 B_i f^i$$



with real coefficients  $A_i, B_i$  such that  $\varphi$  and  $\psi$  have zero of order 2 at  $p$ . It follows that

$$\varphi(p) = \varphi_z(p) = \varphi_{\bar{z}}(p) = 0, \quad \psi(p) = \psi_z(p) = \psi_{\bar{z}}(p) = 0.$$

As it was remarked before, the equations (15), (16), (17) hold for  $\varphi$  and  $\psi$ . It follows that they are eigenfunctions with eigenvalue  $\lambda_2 = 2$  with zero of order 3 at the point  $p$ . This contradicts Lemma 4.5.

It is proven at this moment that for any smooth extremal metric  $g$  (possibly with conical singularities) one has

$$\bar{\lambda}_2(\mathbb{RP}^2, g) = 12\pi < 20\pi.$$

This means that the supremum  $\Lambda_2(\mathbb{RP}^2) = \sup \bar{\lambda}_2(\mathbb{RP}^2, g)$  is obtained in the case I on the projective plane and the sphere with standard metrics touching in a point such that the ratio of areas of the projective plane and the sphere is 3 : 2. This finishes the proof.  $\square$

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(Nikolai S. Nadirashvili) CNRS, I2M UMR 7353 — CENTRE DE MATHÉMATIQUES ET INFORMATIQUE, MARSEILLE, FRANCE

*E-mail address:* [nikolay.nadirashvili@univ-amu.fr](mailto:nikolay.nadirashvili@univ-amu.fr)

(Alexei V. Penskoi) DEPARTMENT OF HIGHER GEOMETRY AND TOPOLOGY, FACULTY OF MATHEMATICS AND MECHANICS, MOSCOW STATE UNIVERSITY, LENINSKIE GORY, GSP-1, 119991, MOSCOW, RUSSIA

and

FACULTY OF MATHEMATICS, NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, 6 USACHEVA STR., 119048, MOSCOW, RUSSIA

and

LABORATOIRE J.-V.PONCELET (UMI 2615), BOLSHOY VLASYEVSKIY PEREULOK 11, 119002, MOSCOW, RUSSIA

*E-mail address, Corresponding author:* [penskoi@mccme.ru](mailto:penskoi@mccme.ru)